



Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa



Generalized matrix version of reverse Hölder inequality

Rupinderjit Kaur ^{a,*}, Mandeep Singh ^a, Jaspal Singh Aujla ^b

^a Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India

^b Department of Mathematics, National Institute of Technology, Jalandhar 144011, Punjab, India

ARTICLE INFO

Article history:

Received 13 July 2010

Accepted 9 September 2010

Available online 13 October 2010

Submitted by C.K. Li

AMS classification:

15A18

15A42

15A60

47A30

47A63

47A64

Keywords:

Positive definite matrices

Matrix means

Positive linear maps

Hölder/Cauchy–Schwarz reverse inequality

ABSTRACT

A generalized matrix version of reverse Cauchy–Schwarz/Hölder inequality is proved. This includes the recent results proved by Bourin, Fujii, Lee, Niezgoda and Seo.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

In what follows, the capital letters A, B, C, \dots denote the $n \times n$ (n arbitrary but fixed) matrices over the algebra of complex numbers, i.e. elements of $M_n(\mathbb{C})$. By $A \geq B$ ($A > B$) we mean that $A - B$ is positive semidefinite (positive definite). A linear map $\phi: M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ is called positive if it maps positive definite matrices into positive definite matrices. We use the notation $\langle \cdot, \cdot \rangle$ to denote the usual inner product in \mathbb{C}^n and $\| \cdot \|$ denotes the vector norm of a vector in \mathbb{C}^n with respect to this inner

* Corresponding author.

E-mail addresses: rupinder_grewal_86@yahoo.co.in (R. Kaur), msrawla@yahoo.com (M. Singh), aujlaajs@yahoo.com (J. Singh Aujla).

product. It is well known that $M_n(\mathbb{C})$ is an inner product space where the canonical inner product is given by $(A, B) = \text{Tr}(AB^*)$ (Here $\text{Tr}(X)$ denotes trace of X).

The axiomatic theory for connections and means of pairs of positive matrices have been developed by Nishio and Ando [8] and Kubo and Ando [5]. A binary operation σ defined on the set of positive definite matrices is called a connection if

- (i) $A \leq C, B \leq D$ implies $A\sigma B \leq C\sigma D$,
- (ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,
- (iii) $A_k \downarrow A$ and $B_k \downarrow B$ imply $A_k\sigma B_k \downarrow A\sigma B$.

A mean is a connection with normalization condition,

- (iv) $I\sigma I = I$.

Kubo and Ando [5] proved the existence of an affine order isomorphism between the class of connections and the class of positive operator monotone functions on \mathbb{R}^+ . This isomorphism $\sigma \leftrightarrow f$ is characterized by the relation

$$A\sigma B = A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

The operator means corresponding to operator monotone functions $t^\alpha, 0 < \alpha < 1$ are called weighted geometric means and are denoted by $\#_\alpha$. In particular $\#_{1/2}$ is called geometric mean.

In Section 2 we shall prove general results which unify and include the recent results proved by Bourin, Fujii, Lee, Niezgoda and Seo in [3,4,6,7].

2. Main results

We begin this section by stating our main results.

Theorem 1. Let $A, B > 0$ be such that $aA \geq B \geq bA$ for some scalars $0 < b \leq a$ and let ϕ be a positive linear map. Then for any connection σ ,

$$\phi(A)\sigma\phi(B) \leq \frac{1}{\omega}\phi(A\sigma B),$$

where $\omega = \frac{f(a)-f(b)}{(a-b)f'(c)}$ for some fixed $c \in (b, a)$ and f is the representing function of σ .

Theorem 2. Let $A, B > 0$ be such that $aA \geq B \geq bA$ for some scalars $0 < b \leq a$ and let ϕ be a positive linear map. Then for any connection σ ,

$$\phi(A)\sigma\phi(B) - \phi(A\sigma B) \leq -g(t_0)\phi(A),$$

where $g(t) = \mu t + \nu - f(t)$, t_0 a fixed point in (b, a) and f is the representing function of σ , $\mu = \frac{f(a)-f(b)}{a-b}$ and $\nu = \frac{af(b)-bf(a)}{a-b}$.

To prove Theorems 1 and 2, we need the following lemmas.

Lemma 3. Let $Z > 0$ be such that $aI \geq Z \geq bI$ for some scalars $0 < b \leq a$ and let $h \in \mathbb{C}^n$ be a unit vector. Then

$$\omega f(\langle Zh, h \rangle) \leq \langle f(Z)h, h \rangle,$$

where $\omega = \frac{f(a)-f(b)}{(a-b)f'(c)}$, $c \in (b, a)$ and f is a positive operator monotone function on \mathbb{R}^+ .

Proof. First notice that the result is trivially true if $b = a$. Indeed $a \rightarrow b$ implies $c \rightarrow b$ and hence $\omega \rightarrow 1$. Moreover, Z reduces to a scalar matrix in this case and so $f(\langle Zh, h \rangle) = \langle f(Z)h, h \rangle = f(b)$. We may therefore assume without loss of generality that $0 < b < a$.

Consider the line $\mu t + \nu$ where $\mu = \frac{f(a)-f(b)}{a-b}$ and $\nu = \frac{af(b)-bf(a)}{a-b}$. Note that $f(t)$ and the line $\mu t + \nu$ intersect at the points $(a, f(a))$ and $(b, f(b))$. f being operator monotone, is concave and strictly increasing on \mathbb{R}^+ . Thus, using the concavity of $f(t)$ we see that

$$\mu t + v \leq f(t) \quad (1)$$

for all $t \in [b, a]$. By Lagrange's Mean Value Theorem there exists a point $d \in (b, a)$ with $f'(d) = \frac{f(a)-f(b)}{a-b}$. We can write $d = \beta a + (1 - \beta)b$ for some $\beta \in (0, 1)$. Let

$$F_\alpha(t) = \mu t + v - \omega_\alpha f(t),$$

where $\omega_\alpha = \frac{f(a)-f(b)}{(a-b)f'(\alpha a + (1-\alpha)b)}$ for $0 \leq \alpha \leq 1$. $F_\alpha(t)$ is a convex function of t with minima at $t = \alpha a + (1 - \alpha)b$, since $F'_\alpha(\alpha a + (1 - \alpha)b) = 0$. Thus $F_0(t)$ attains its minimum at $t = b$ in $[b, a]$. Note that $F_0(b) = f(b)(1 - \omega_0)$ is non-negative. This follows from the facts that $f'(t)$ is decreasing in $[b, a]$ and so $f'(b) \geq f'(d)$ which imply

$$\omega_0 = \frac{f(a) - f(b)}{(a - b)f'(b)} \leq \frac{f(a) - f(b)}{(a - b)f'(d)} = 1.$$

Thus $F_0(t) \geq 0$ for all $t \in [b, a]$. Further note that $\omega_\beta = 1$, and therefore it follows from (1) that $F_\beta(t) \leq 0$ for all $t \in [b, a]$. Since the function $\alpha \rightarrow F_\alpha$ is continuous, it follows that there exists at least one $\alpha \in (0, 1)$ such that $F_\alpha(t) \geq 0$ for all $t \in [b, a]$. Let $\alpha_0 = \sup\{\alpha \in (0, 1) : F_\alpha(t) \geq 0 \text{ for all } t \in [b, a]\}$. Choose $c = \alpha_0 a + (1 - \alpha_0)b$ and $\omega = \omega_{\alpha_0} = \frac{f(a)-f(b)}{(a-b)f'(c)}$. Then $F_{\alpha_0}(t) \geq 0$ for all $t \in [b, a]$. This implies

$$\omega f(t) \leq \mu t + v \quad (2)$$

for all $t \in [b, a]$. Let $t = \langle Zh, h \rangle$. Since $aI \geq Z \geq bI$ and h is a unit vector, we have $a \geq h^*Zh \geq b$, i.e., $a \geq t \geq b$. Therefore putting this t in (2) we have

$$\omega f(\langle Zh, h \rangle) \leq \mu \langle Zh, h \rangle + v \leq \langle f(Z)h, h \rangle.$$

The second inequality in the above inequality follows from inequality (1). This completes the proof. \square

Lemma 4. Let $A, B > 0$ be such that $aA \geq B \geq bA$ for some scalars $0 < b \leq a$ and let $h \in \mathbb{C}^n$ be a unit vector. Then

$$\langle (A\sigma B)h, h \rangle \leq \langle Ah, h \rangle \sigma \langle Bh, h \rangle \leq \omega^{-1} \langle (A\sigma B)h, h \rangle$$

for all connections σ with representing function f and ω is as in Lemma 3.

Proof. Let $x \in \mathbb{C}^n$, $x \neq 0$ and $h = \frac{x}{\|x\|}$. Let $Z = A^{-1/2}BA^{-1/2}$. Since $aA \geq B \geq bA$ which implies $aI \geq A^{-1/2}BA^{-1/2} \geq bI$, so $aI \geq Z \geq bI$. By concavity of $f(t)$, we have

$$\left\langle f(Z) \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \leq f \left(\left\langle Z \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \right) \quad (3)$$

(see [2, p. 281]). On using Lemma 3, we have

$$f \left(\frac{1}{\|x\|^2} \langle Zx, x \rangle \right) \leq \frac{\omega^{-1}}{\|x\|^2} \langle f(Z)x, x \rangle. \quad (4)$$

Combining (3) and (4) we obtain,

$$\frac{1}{\|x\|^2} \langle f(Z)x, x \rangle \leq f \left(\frac{1}{\|x\|^2} \langle Zx, x \rangle \right) \leq \frac{\omega^{-1}}{\|x\|^2} \langle f(Z)x, x \rangle,$$

or equivalently

$$\langle f(Z)x, x \rangle \leq \|x\|^2 f \left(\frac{1}{\|x\|^2} \langle Zx, x \rangle \right) \leq \omega^{-1} \langle f(Z)x, x \rangle. \quad (5)$$

Replacing Z by $A^{-1/2}BA^{-1/2}$ and x by $A^{1/2}h$, in (5) we get,

$$\begin{aligned} \langle A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}h, h \rangle &\leq \langle Ah, h \rangle f(\langle Ah, h \rangle^{-1} \langle Bh, h \rangle) \\ &\leq \omega^{-1} \langle A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}h, h \rangle, \end{aligned}$$

i.e.

$$\langle (A\sigma B)h, h \rangle \leq \langle Ah, h \rangle \sigma \langle Bh, h \rangle \leq \omega^{-1} \langle (A\sigma B)h, h \rangle.$$

This gives the desired inequality. \square

We need the next lemma to prove Theorem 2.

Lemma 5. Let $A, B > 0$ be such that $aA \geq B \geq bA$ for some scalars $0 < b \leq a$, $h \in \mathbb{C}^n$ be a unit vector and σ be a connection. Then

$$\langle Ah, h \rangle \sigma \langle Bh, h \rangle - \langle (A\sigma B)h, h \rangle \leq -g(t_0) \langle Ah, h \rangle, \quad (6)$$

where f is the representing function of σ , $g(t) = \mu t + \nu - f(t)$, where $\mu = \frac{f(a)-f(b)}{a-b}$, $\nu = \frac{af(b)-bf(a)}{a-b}$ and $t_0 \in (b, a)$.

Proof. Again, as in Lemma 3, we may take without loss of generality that $0 < b < a$, since $a \rightarrow b$ implies $t_0 \rightarrow b$ and hence $g(t_0) \rightarrow 0$, while on using the definition of σ a connection, the left side of the inequality (6) approaches to 0.

Consider the line $\mu t + \nu$ where $\mu = \frac{f(a)-f(b)}{a-b}$ and $\nu = \frac{af(b)-bf(a)}{a-b}$. As in the proof of Lemma 3 we have

$$\mu t + \nu \leq f(t)$$

for all $t \in [b, a]$. By Lagrange's Mean Value Theorem there exists $t_0 \in (b, a)$ such that $f'(t_0) = \frac{f(a)-f(b)}{a-b} = \mu$. Let

$$F(t) = \mu t + \nu - f(t) - g(t_0).$$

This is a convex function of t . Thus we see that $F'(t_0) = 0$. This along with the convexity of $F(t)$ imply that $F(t)$ has minimum value at $t = t_0$. Further note that $F(t_0) = 0$. Thus $F(t) \geq 0$ for all $t \in [b, a]$. Hence

$$f(t) - (\mu t + \nu) \leq -g(t_0), \quad t \in [b, a]. \quad (7)$$

Let $Z = A^{-1/2}BA^{-1/2}$ and $x \in \mathbb{C}^n, x \neq 0$. Taking $t = \left\langle A^{-1/2}BA^{-1/2} \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \in [b, a]$, in (7) we obtain

$$\|x\|^2 f\left(\frac{1}{\|x\|^2} \langle A^{-1/2}BA^{-1/2}x, x \rangle\right) - \left\langle f\left(A^{-1/2}BA^{-1/2}\right)x, x \right\rangle \leq -g(t_0)\|x\|^2,$$

and hence,

$$\langle x, x \rangle \sigma \langle A^{-1/2}BA^{-1/2}x, x \rangle - \left\langle f\left(A^{-1/2}BA^{-1/2}\right)x, x \right\rangle \leq -g(t_0)\langle x, x \rangle. \quad (8)$$

Now if we replace x by $A^{1/2}h$, in (8) we get the desired inequality. \square

Proof of Theorem 1. First suppose that ϕ is given by $\phi(A) = \langle Ah, h \rangle$ for any vector $h \in \mathbb{C}^n$. Then by Lemma 4,

$$\begin{aligned} \phi(A)\sigma\phi(B) &= \langle Ah, h \rangle \sigma \langle Bh, h \rangle \\ &\leq \omega^{-1} \langle (A\sigma B)h, h \rangle \\ &\leq \omega^{-1} \phi(A\sigma B), \end{aligned}$$

which proves the theorem in this case.

Now consider the case of general positive linear map ϕ . Let $h \in \mathbb{C}^n$ be any vector. Then, by first inequality of Lemma 4,

$$\langle (\phi(A)\sigma\phi(B))h, h \rangle \leq \langle \phi(A)h, h \rangle \sigma \langle \phi(B)h, h \rangle = \psi(A)\sigma\psi(B) \quad (9)$$

where $\psi(A) = \langle \phi(A)h, h \rangle$.

Note that ψ is a positive linear functional on $M_n(\mathbb{C})$. Therefore there exists $X \geq 0$ such that $\psi(A) = \text{Tr}AX$. Hence, if $\pi(A) : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is left multiplication by A , then

$$\psi(A) = (\pi(A)h, h)$$

where $h = X^{1/2}$. Since $aA \geq B \geq bA$ implies $a\pi(A) \geq \pi(B) \geq b\pi(A)$, the second inequality of Lemma 4 yields

$$\psi(A)\sigma\psi(B) \leq \omega^{-1}\psi(A\sigma B) \quad (10)$$

Combining (9) and (10) we have,

$$\phi(A)\sigma\phi(B) \leq \frac{1}{\omega}\phi(A\sigma B). \quad \square$$

Proof of Theorem 2. The proof of this theorem is similar to the proof of Theorem 1 on using Lemma 5 and is therefore not included. \square

Remark 6. The main results proved in [3,6,7] follow by taking $\sigma = \#_\alpha$ in Theorem 1 and main results proved in [4] follow on taking $\sigma = \#_\alpha$ in Theorem 2. It is further remarked that the inequality

$$\phi(A\sigma B) \leq \phi(A)\sigma\phi(B)$$

is proved in [1].

Acknowledgement

The authors like to thank a referee for useful comments.

References

- [1] J.S. Aujla, H.L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japan.* 42 (1995) 265–272.
- [2] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
- [3] J.-C. Bourin, E.-Y. Lee, M. Fujii, A matrix reverse Hölder inequality, *Linear Algebra Appl.* 431 (2009) 2154–2159.
- [4] M. Fujii, E.-Y. Lee, Y. Seo, A difference counterpart to matrix Hölder inequality, *Linear Algebra Appl.* 432 (2010) 2565–2571.
- [5] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.* 246 (1980) 205–224.
- [6] E.-Y. Lee, A matrix reverse Cauchy–Schwarz inequality, *Linear Algebra Appl.* 430 (2009) 805–810.
- [7] M. Niezgoda, Accretive operators and Cassels inequality, *Linear Algebra Appl.* 433 (2010) 136–142.
- [8] K. Nishio, T. Ando, Characterizations of operations derived from network connections, *J. Math. Anal. Appl.* 53 (1976) 539–549.